*J. Aust. Math. Soc.* **95** (2013), 315–328
doi:10.1017/S1446788713000335

# PARABOLIC HIGGS BUNDLES AND Γ-HIGGS BUNDLES

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(Received 31 August 2012; accepted 5 June 2013; first published online 19 August 2013)

Communicated by D. Chan

**Abstract**

We investigate parabolic Higgs bundles and Γ-Higgs bundles on a smooth complex projective variety.

2010 *Mathematics subject classification*: primary 14D23; secondary 14H60.
*Keywords and phrases*: parabolic Higgs bundle, root stack, Γ Higgs bundle.

## 1. Introduction

Let $X$ be a compact connected Riemann surface and $E$ a holomorphic vector bundle on $X$. The infinitesimal deformations of $E$ are parametrized by $H^1(X, \mathrm{End}\, E)$, where $\mathrm{End}\ E = E \otimes E^*$ is the sheaf of endomorphisms of the vector bundle $E$. By Serre duality, we have $H^1(X, \mathrm{End}\, E)^* = H^0(X, (\mathrm{End}\, E) \otimes \Omega^1_X)$, where $\Omega^1_X$ is the holomorphic cotangent bundle of $X$. A Higgs field on $E$ is defined to be a holomorphic section of $(\mathrm{End}\, E) \otimes \Omega^1_X$; they were introduced by Hitchin [Hi87a, Hi87b]. A Higgs bundle is a holomorphic vector bundle equipped with a Higgs field. Hitchin proved that stable Higgs bundles of rank $r$ and degree zero on $X$ are in bijective correspondence with the irreducible flat connections on $X$ of rank $r$ [Hi87a]. He also proved that the moduli space of Higgs bundles on $X$ of rank $r$ is a holomorphic symplectic manifold, and the space of holomorphic functions on this holomorphic symplectic manifold gives it the structure of an algebraically completely integrable system [Hi87b]. Simpson arrived at Higgs bundles via his investigations of variations of Hodge structures [Si88]. He extended the results of Hitchin to Higgs bundles over higher dimensional complex projective manifolds.

A parabolic structure on a holomorphic vector bundle $E$ on $X$ is roughly a system of weighted filtrations of the fibers of $E$ over some finitely many given points. A parabolic vector bundle is a holomorphic vector bundle equipped with a parabolic

MLW would like to express his gratitude to the Fonds québécois de la recherche sur la nature et les technologies for the support of a Bourse de recherche postdoctorale (B3).

structure; parabolic vector bundles were introduced by Mehta and Seshadri [MS80]. Parabolic vector bundles with Higgs structure were introduced by Yokogawa [Yo95].

Our aim here is to investigate the parabolic vector bundles equipped with a Higgs structure. More precisely, we study the relationship between the parabolic Higgs bundles and the Higgs vector bundles on a root stack.

Root stacks are important examples of smooth Deligne–Mumford stacks; see [Cad07, Bo07] for root stacks.

Let $Y$ be a smooth complex projective variety on which a finite group $\Gamma$ acts as a group of automorphisms satisfying the condition that the quotient $X = \Gamma \setminus Y$ is also a smooth variety. There is a bijective correspondence between the *parabolic Higgs bundles* on $X$ and the $\Gamma$-*Higgs bundles* on $Y$. We prove that the parabolic Higgs bundles on $X$ are identified with the Higgs bundles on the associated root stack.

The organization of the paper is as follows. In Section 3 we review the notions of parabolic bundle and $\Gamma$-vector bundle and define the parabolic Higgs bundle and $\Gamma$-Higgs bundles. As was done for parabolic vector bundles in [Bi97] we describe an equivalence between the category of $\Gamma$-Higgs bundles on $Y$ and parabolic Higgs bundle on $X$.

Section 4.1 describes the construction of a root stack as done in [Cad07]. In Section 4.2, we investigate vector bundles on root stacks. Section 4.3 generalizes Theorem 3.5 to the case of the root stack over the base space.

## 2. Preliminaries

Let $Y$ be a smooth complex projective variety of dimension $m$ endowed with the action $\lambda : \Gamma \times Y \longrightarrow Y$ of a finite group $\Gamma$ such that:

(1) $X := \Gamma \setminus Y$ is also a smooth variety; and
(2) the projection map $\pi : Y \longrightarrow X$ is a Galois covering with Galois group $\Gamma$.

The closed subset of $Y$ consisting of points with nontrivial isotropy subgroups for the action of $\Gamma$ is a divisor $\widehat{D} \subset Y$ [Bi97, Lemma 2.8]. Let $D \subset X$ be its (reduced) image under $\pi$ and $D = D_1 + \cdots + D_h$ its decomposition into irreducible components. We will always be working with the assumption that the $D_\mu$, $1 \le \mu \le h$, are smooth, and $D$ is a normal crossing divisor; this means that all the intersections of the irreducible components of $D$ are transversal. For an effective divisor $D'$ on $Y$, by $D'_{\mathrm{red}}$ we will denote the corresponding reduced divisor. So $D'_{\mathrm{red}}$ is obtained from $D'$ by setting all the multiplicities to be one. We set

$$\widetilde{D}_\mu := (\pi^* D_\mu)_{\mathrm{red}}, \quad 1 \le \mu \le h, \quad \widetilde{D} := \sum_{\mu=1}^{h} \widetilde{D}_\mu.$$

There exist $k_\mu, r \in \mathbb{N}$ for $1 \le \mu \le h$ such that $\pi^* D_\mu = k_\mu r \widetilde{D}_\mu$; with this,

$$\pi^* D = r \sum_{\mu=1}^{h} k_\mu \widetilde{D}_\mu.$$

It should be clarified that there are many choices of $k_\mu$ and $r$. We will write $\overline{D} := \sum_{\mu=1}^{h} k_\mu \widetilde{D}_\mu$, so that $\pi^* D = r\overline{D}$.

LEMMA 2.1. *With the assumption that $D$ is a normal crossing divisor with smooth components, given a point $y \in Y$ with $\pi(y) \in \bigcap_1^l D_\mu$, one can choose coordinates $w_1, \ldots, w_m$ in an analytic neighborhood of $y$ and $z_1, \ldots, z_m$ in an analytic neighborhood of $\pi(y)$ such that $D_\mu$ is defined by $z_\mu$ for $1 \le \mu \le l$, $\widetilde{D}_\mu$ is defined by $w_\mu$ for $1 \le \mu \le l$ and $\pi$ is given in the local coordinates by*

$$z_1 = w_1^{k_1 r}, \ldots, z_l = w_l^{k_l r}, \quad z_{l+1} = w_{l+1}, \ldots, z_m = w_m.$$

PROOF. This is proved in [Na, page 11, Theorem 1.1.14]. □

Recall [Del70, Section II.3] that the sheaf $\Omega_X^1(\log D)$ of logarithmic differentials with poles at $D$ is the locally free sheaf on $X$, a basis for which in the neighborhood of a point in $\bigcap_1^l D_\mu$ with coordinates chosen as in the lemma is given by

$$\frac{dz_\mu}{z_\mu}, 1 \le \mu \le l, \quad dz_\mu, l+1 \le \mu \le m.$$

Therefore, the dual $\Omega_X^1(\log D)^*$ is the subsheaf of the holomorphic tangent bundle $TX$ given by the sheaf of vector fields that preserve $O_X(-D) \subset O_X$.

We have the following analogue of Hurwitz's theorem.

LEMMA 2.2. *With $\pi : Y \longrightarrow X$ as above, one has*

$$\Omega_Y^1(\log \widetilde{D}) \cong \pi^* \Omega_X^1(\log D).$$

PROOF. This follows immediately from [EV92, page 33, Lemma 3.21]. □

Observe that we have inclusions of sheaves

$$\Omega_X^1 \subseteq \Omega_X^1(\log D) \subseteq \Omega_X^1(D) := \Omega_X^1 \otimes O_X(D),$$
$$\Omega_Y^1 \subseteq \Omega_Y^1(\log \widetilde{D}) \subseteq \Omega_Y^1(\widetilde{D}) := \Omega_Y^1 \otimes O_Y(\widetilde{D}).$$

Fixing an irreducible component $D_\mu$ of $D$, there is a residue map (see [Del70, Section II.3.7])

$$\mathrm{Res}_{D_\mu} : \Omega_X^1(\log D_\mu) \longrightarrow \mathscr{O}_{D_\mu}.$$

In local coordinates $z_1, \ldots, z_m$ on $X$ where $D_\mu$ is defined by $z_\mu = 0$, if $\omega$ is a section of $\Omega_X^1(\log D_\mu)$ with local expression

$$\omega = f_1\, dz_1 + \cdots + f_\mu \frac{dz_\mu}{z_\mu} + \cdots + f_m\, dz_m,$$

where the $f_i$, $1 \le i \le m$, are holomorphic functions, then the residue has the local expression

$$\mathrm{Res}_{D_\mu} \omega = f_\mu|_{z_\mu = 0}.$$

## 3. Parabolic Higgs bundles and Γ-Higgs bundles

**3.1. Parabolic Higgs bundles.** Let $E$ be a torsion-free coherent sheaf on $X$. We recall that a *parabolic structure* on $E$ with respect to the divisor $D$ is the data of a filtration

$$E = E_{\alpha_1} \supset E_{\alpha_2} \supset \cdots \supset E_{\alpha_l} \supset E_{\alpha_{l+1}} = E(-D),$$

where $0 \leq \alpha_1 < \cdots < \alpha_l < 1$ are real numbers called *weights* (see [MY92, Definition 1.2]). The $\alpha_j$ will be chosen without redundancy in the sense that if $\epsilon > 0$, then $E_{\alpha_j+\epsilon} \neq E_{\alpha_j}$. We will often shorten $E_{\alpha_j}$ to $E_j$. The sheaf $E$ together with a parabolic structure is called a *parabolic sheaf* and is often denoted by $E_*$. If $E$ is a locally free sheaf, then we will call $E_*$ a *parabolic vector bundle*. See [MY92] for more on parabolic sheaves.

We will always assume that the parabolic weights are rational numbers whose denominators all divide $r \in \mathbb{N}$, that is, $\alpha_j \in (1/r)\mathbb{Z}$, for $1 \leq j \leq l$; this way, we may write $\alpha_j = m_j/r$ for some integers $0 \leq m_j \leq r-1$. It should be clarified that there are many choices for $r$. Further, we will make the same assumptions as in [Bi97, Assumptions 3.2].

A *parabolic Higgs field*, respectively *strongly parabolic Higgs field*, will be defined as a section $\phi \in H^0(X, (\operatorname{End} E) \otimes \Omega^1_X(\log D))$ satisfying

$$\phi \wedge \phi = 0$$

and

$$(\operatorname{Res}_{D_\mu}\phi)(E_j|_{D_\mu}) \subseteq E_j|_{D_\mu}, \text{ respectively } (\operatorname{Res}_{D_\mu}\phi)(E_j|_{D_\mu}) \subseteq E_{j+1}|_{D_\mu}, \tag{3.1}$$

for $1 \leq j \leq l$, $1 \leq \mu \leq h$. By a *parabolic Higgs bundle* we will mean a pair $(E_*, \phi)$ consisting of a parabolic vector bundle $E_*$ and a strongly parabolic Higgs field $\phi$.

Remark 3.1. Observe that this definition of a parabolic Higgs field differs from that given in [Yo95, Definition 2.2], where one takes $\phi \in H^0(X, (\operatorname{End} E) \otimes \Omega^1_X(D))$.

**3.2. Γ-Higgs bundles.** Let $W$ be a vector bundle on $Y$ admitting an action $\Lambda : \Gamma \times W \longrightarrow W$ compatible with the action $\lambda$ on $Y$. If we think of $W$ as a space with projection $r : W \longrightarrow Y$, then this means that

$$\begin{tikzcd} \Gamma \times W \arrow[r, "\Lambda"] \arrow[d, "\mathbb{1}_\Gamma \times r"'] & W \arrow[d, "r"] \\ \Gamma \times Y \arrow[r, "\lambda"'] & Y \end{tikzcd}$$

commutes. Alternatively, if we think of $W$ as a locally free sheaf then this means that there is an isomorphism

$$L : \lambda^* W \xrightarrow{\sim} p_Y^* W$$

of sheaves on $\Gamma \times Y$ satisfying a suitable cocycle condition. When such an action exists, we will call $W$ a $\Gamma$-*vector bundle*. In this realization, if $W'$ is another $\Gamma$-vector bundle with $L' : \lambda^* W' \longrightarrow p_Y^* W'$ giving the action on $W'$, then compatible actions on $W \oplus W'$ and $W \otimes W'$ are readily defined since direct sums and tensor products commute with pullbacks.

For each $\gamma \in \Gamma$, the restrictions $L_{\{\gamma\}\times Y} : \lambda_\gamma^* W \xrightarrow{\sim} W$ yield isomorphisms $L_\gamma : W \longrightarrow \lambda_{\gamma *} W$ (by adjunction) satisfying

$$L_e = \mathbb{1}_W \quad \text{and} \quad \lambda_{\gamma *} L_\delta \circ L_\gamma = L_{\gamma\delta}$$

for all $\gamma, \delta \in \Gamma$. In our case, since $\Gamma$ is discrete, knowledge of the $L_\gamma$ is enough to reconstruct $L$.

Example 3.2. There are three examples of Γ-bundles that will be of particular interest to us.

(a) The action $\lambda$ on $Y$ induces a natural action on the sheaf of differentials $\Omega_Y^1$ which will be compatible with $\lambda$.

(b) Since $X = \Gamma \setminus Y$, we have $\pi \circ \lambda = \pi \circ p_Y$ as maps $\Gamma \times Y \longrightarrow X$. Thus, if $E$ is any vector bundle on $X$, there is a canonical isomorphism $\lambda^* \pi^* E \xrightarrow{\sim} p_Y^* \pi^* E$. Hence the pullback $\pi^* E$ carries a Γ-action for which the action on the fibers is induced by the action on $Y$.

(c) By the previous example, $\mathscr{O}_Y(\pi^* D) = \pi^* \mathscr{O}_X(D)$ carries a compatible Γ-action. Since $\widetilde{D} \subseteq \pi^* D$ is a Γ-invariant subset we have an induced action on the line bundle $\mathscr{O}_Y(\widetilde{D})$ making it into a Γ-line bundle.

Let $W, W'$ be as above. A homomorphism $\Phi : W \longrightarrow W'$ is said to *commute with the* Γ-*actions* or is a Γ-*homomorphism* if the diagram

$$\begin{array}{ccc} \lambda^* W & \xrightarrow{\lambda^* \Phi} & \lambda^* W' \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L'} \\ p_Y^* W & \xrightarrow[p_Y^* \Phi]{} & p_Y^* W' \end{array}$$

commutes.

If $\Phi \in H^0(Y, (\mathrm{End}\, W) \otimes \Omega_Y^1)$ is a Higgs field on $W$, that is, $\Phi \wedge \Phi = 0$, then we will call it a Γ-*Higgs field* if as a map $W \longrightarrow W \otimes \Omega_Y^1$ it commutes with the Γ-actions, where $W \otimes \Omega_Y^1$ has the tensor product action. Thus, for every $\gamma \in \Gamma$, there is a commutative diagram.

$$\begin{array}{ccc} W & \xrightarrow{\Phi} & W \otimes \Omega_Y^1 \\ {\scriptstyle L_\gamma}\downarrow & & \downarrow{\scriptstyle \widetilde{L}_\gamma} \\ \lambda_{\gamma *} W & \xrightarrow[\lambda_{\gamma *} \Phi]{} & \lambda_{\gamma *}(W \otimes \Omega_Y^1) \end{array} \tag{3.2}$$

If $\Phi$ is a Γ-Higgs field, the pair $(W, \Phi)$ will be referred to as a Γ-*Higgs bundle*.

**3.3. From Γ-Higgs bundles to parabolic Higgs bundles.** We now begin with a Γ-Higgs bundle $(W, \Phi)$ and from it construct a parabolic Higgs bundle $(E_*, \phi)$. The underlying vector bundle $E$ is defined as $E := \pi_* W^\Gamma$, the sheaf of Γ-invariant sections of $\pi_* W$, and as in [Bi97, Section 2c], the parabolic structure on $E$ is defined by

$$E_j := \pi_* W\Big(\sum_{\mu=1}^{h} \lfloor -k_\mu r \alpha_j \rfloor \widetilde{D}_\mu\Big)^\Gamma.$$

Suppose $\Phi \in H^0(Y, (\mathrm{End}\, W) \otimes \Omega^1_Y)$ is a Γ-Higgs field on $W$. We will think of $\Phi$ as a homomorphism $\Phi : W \longrightarrow W \otimes \Omega^1_Y$. Since $\Omega^1_Y \subseteq \Omega^1_Y(\log \widetilde{D})$,

$$\pi_*(W \otimes \Omega^1_Y) \subseteq \pi_*(W \otimes \Omega^1_Y(\log \widetilde{D})) = \pi_*(W \otimes \pi^*\Omega^1_X(\log D)) = \pi_* W \otimes \Omega^1_X(\log D),$$

where the first equality is due to Lemma 2.2 and the last step by the projection formula. Therefore, $\phi := \pi_*\Phi$ may be considered as a map $\pi_* W \longrightarrow \pi_* W \otimes \Omega^1_X(\log D)$, and we have a candidate for a parabolic Higgs field.

Let $U \subseteq X$ be open and let $s$ be an invariant section of $\pi_* W$ over $U$, so that we may think of $s$ as a section $\widehat{s}$ of $W$ over $\pi^{-1}(U)$ with $L_\gamma \widehat{s} = \widehat{s}$ for all $\gamma \in \Gamma$. Then by definition $\phi s := \Phi\widehat{s}$, and for $\gamma \in \Gamma$, using (3.2),

$$\widetilde{L}_\gamma(\phi s) = \widetilde{L}_\gamma(\Phi\widehat{s}) = \Phi(L_\gamma \widehat{s}) = \Phi\widehat{s} = \phi s,$$

so $\phi s$ is a Γ-invariant section, and hence $\phi : E \longrightarrow E \otimes \Omega^1_X(\log D)$.

Proposition 3.3. *To a Γ-Higgs bundle $(W, \Phi)$ there is a naturally associated parabolic Higgs bundle $(E_*, \phi)$.*

Proof. We have constructed $(E_*, \phi)$. We must prove that $\phi$ is strongly parabolic. This is a condition on the residues of $\phi$ along the components of the divisor $D$, so we may concentrate on those points of $D_\mu$ that do not belong to any other component of $D$. Therefore, we may assume that we are in the neighborhood of a point $y$ of $\widetilde{D}_1$ that lies on no other $\widetilde{D}_\mu$. In this neighborhood, for $1 \le j \le l$,

$$E_j = \pi_* W(-m_j k_1 \widetilde{D}_1)^\Gamma.$$

We now choose coordinates on $Y$ and $X$ as in Lemma 2.1, so that the divisor $\widetilde{D}_1$ is defined by $w_1$ and the divisor $D_1$ is defined by $z_1$; we will write $p := k_1 r$ so that $\pi$ is given in these coordinates by

$$z_1 = w_1^p, \quad z_2 = w_2, \ldots, z_m = w_m.$$

In these coordinates, near $y$, we may write

$$\Phi = A_1 dw_1 + \cdots + A_m dw_m$$

for some holomorphic sections $A_i$ of End $W$. We may then consider $A_1 dw_1 = (1/p)A_1 w_1 dz_1/z_1$ as a locally defined map $W \longrightarrow W \otimes \mathscr{O}_Y(-\widetilde{D}_1) \otimes \pi^*\Omega^1_X(\log D_1)$, or more generally, as a map

$$W(-m_j k_1 \widetilde{D}_1) \longrightarrow W(-(m_j k_1 + 1)\widetilde{D}_1) \otimes \pi^*\Omega^1_X(\log D_1)$$

for $1 \le j \le l$. It is easily verified that $W(-(m_j k_1 + 1)\widetilde{D}_1) \subseteq W(-rk_1(\alpha_j + \epsilon)\widetilde{D}_1)$, where $0 \le \epsilon \le 1/rk_1$. So taking invariants we see that $\pi_* A_1 dw_1 = (1/p)A_1 w_1 dz_1/z_1$ gives a locally defined map

$$E_j \longrightarrow E_{\alpha_j + (1/rk_1)} \otimes \Omega^1_X(\log D_1) = E_{j+1} \otimes \Omega^1_X(\log D_1).$$

Since, by definition,

$$\mathrm{Res}_{D_1}\phi = \tfrac{1}{p} w_1 A_1|_{z_1=0},$$

and noting that $(\mathrm{Res}_{D_\mu}\phi)(E_j|_{D_1}) \subseteq E_{j+1}|_{D_1}$, it follows that the strong parabolicity condition (3.1) is satisfied.

That $\phi \wedge \phi = 0$ is easily seen, since if $s$ is any section of $E$, then

$$(\phi \wedge \phi)s = (\Phi \wedge \Phi)\widehat{s} = 0$$

since $\Phi \wedge \Phi = 0$. □

**3.4. From parabolic Higgs bundles to Γ-Higgs bundles.** Recall that we are working under assumptions as in [Bi97, Assumptions 3.2], hence we can use the construction from [Bi97, Section 3b] in the following proposition.

Proposition 3.4. *Given a parabolic Higgs bundle* $(E_*, \phi)$ *on* $X$, *we can associate a* $\Gamma$*-Higgs bundle* $(W, \Phi)$ *on* $Y$.

Proof. We will begin by constructing a parabolic vector bundle on $X$ of rank $m$. The holomorphic vector bundle underlying the parabolic vector bundle is $\Omega^1_X$. To define the parabolic structure, take any irreducible component $D_i$ of $D$. Let $\iota : D_i \hookrightarrow X$ be the inclusion map. We have a short exact sequence of vector bundles on $D_i$

$$0 \longrightarrow N^*_{D_i} \longrightarrow \iota^*\Omega^1_X \longrightarrow \Omega^1_{D_i} \longrightarrow 0,$$

where $N_{D_i}$ is the normal bundle of $D_i$. Note that the Poincaré adjunction formula says that $N_{D_i} = \iota^* O_X(D_i)$. The quasiparabolic filtration over $D_i$ is the above filtration

$$N^*_{D_i} \subset \iota^*\Omega^1_X.$$

The parabolic weights are 0 and $(rk_i - 1)/rk_i$. More precisely, $N^*_{D_i}$ has parabolic weight $(rk_i - 1)/rk_i$ and the parabolic weight of the quotient $\Omega^1_{D_i}$ is zero. Note that the nonzero parabolic weight $(rk_i - 1)/rk_i$ has multiplicity one. This parabolic vector bundle will be denoted by $\widetilde{\Omega}^1_X$.

The action of $\Gamma$ on $Y$ induces an action of $\Gamma$ on the vector bundle $\Omega^1_Y$ making it a $\Gamma$-bundle. From the construction of $\widetilde{\Omega}^1_X$ it can be deduced that the parabolic vector bundle corresponding to the $\Gamma$-bundle $\Omega^1_Y$ is $\widetilde{\Omega}^1_X$. To prove this, first note that if $U_0 \subset X$ is a Zariski open subset such that the complement $U_0^c$ is of codimension at least two, and $V$, $W$ are two algebraic vector bundles on $X$ that are isomorphic over $U_0$, then $V$ and $W$ are isomorphic over $X$; using Hartog's theorem, any isomorphism $V|_{U_0} \longrightarrow W|_{U_0}$ extends to a homomorphism $V \longrightarrow W$, and similarly, we have a homomorphism $W \longrightarrow V$, and these two homomorphisms are inverses of each other because they are so over $U_0$. Next note that

$$(\pi_*\Omega^1_Y)^\Gamma = \Omega^1_X$$

because $(\pi_*\Omega^1_Y)^\Gamma = \Omega^1_X$ over the complement of the singular locus of $D$. Therefore, $\Omega^1_X$ is the vector bundle underlying the parabolic bundle corresponding to the $\Gamma$-bundle $\Omega^1_Y$. It is now straightforward to check that the parabolic weights are of the above type.

Let $W$ be the $\Gamma$-bundle on $Y$ corresponding to the parabolic vector bundle $E_*$ on $X$ (using [Bi97, Section 3b]). Let $\phi$ be a strongly parabolic Higgs field on $E_*$. It is straightforward to check that $\phi$ defines a homomorphism of parabolic vector bundles

$$\phi' : E_* \longrightarrow E_* \otimes \widetilde{\Omega}^1_X,$$

where $E_* \otimes \widetilde{\Omega}^1_X$ is the parabolic tensor product of $E_*$ and $\widetilde{\Omega}^1_X$.

Since the correspondence between parabolic bundles and $\Gamma$-vector bundles is compatible with the operation of tensor product, we conclude that the parabolic tensor product $E_* \otimes \widetilde{\Omega}^1_X$ corresponds to the $\Gamma$-bundle $W \otimes \Omega^1_Y$. Therefore, the above homomorphism $\phi'$ pulls back to a $\Gamma$-equivariant homomorphism $\Phi$ from $W$ to $W \otimes \Omega^1_Y$. □

THEOREM 3.5. *We have an equivalence of categories between $\Gamma$-Higgs bundles on $Y$ and parabolic Higgs bundles on $X$ which satisfy the assumptions as in* [Bi97, Assumptions 3.2].

PROOF. Proof is clear from Propositions 3.3 and 3.4 and [Bi97, Sections 2c, 3b]. □

REMARK 3.6. In Borne's formalism, the parabolic bundle $E_*$ may be considered as a functor $((1/r)\mathbb{Z})^{\mathrm{op}} \longrightarrow \mathfrak{Vect}(X)$, with

$$\frac{j}{r} \longmapsto E_j(D),$$

and composing with $\pi^* : \mathfrak{Vect}(X) \longrightarrow \mathfrak{Vect}(Y)$, we get a functor $((1/r)\mathbb{Z})^{\mathrm{op}} \longrightarrow \mathfrak{Vect}(Y)$. We also have a covariant functor $(1/r)\mathbb{Z} \longrightarrow \mathfrak{Vect}(Y)$ given by

$$\frac{j}{r} \longmapsto \mathscr{O}_Y(m_{j-1}\overline{D}).$$

Therefore, we obtain a functor $((1/r)\mathbb{Z})^{\mathrm{op}} \times \frac{1}{r}\mathbb{Z} \longrightarrow \mathfrak{Vect}(Y)$

$$\frac{j}{r} \longmapsto \pi^* E_j(D) \otimes \mathscr{O}_Y(m_{j-1}\overline{D}).$$

An end for this functor [Ma98, Section IX.5] consists of a vector bundle $V \in \operatorname{Ob}\mathfrak{Vect}(Y)$ and diagrams for $i \leq j$

$$\begin{tikzcd} & \pi^* E_i(D) \otimes \mathscr{O}_Y(m_{i-1}\overline{D}) \arrow[dr] & \\ V \arrow[ur] \arrow[dr] & & \pi^* E_i(D) \otimes \mathscr{O}_Y(m_{j-1}\overline{D}) \\ & \pi^* E_j(D) \otimes \mathscr{O}_Y(m_{j-1}\overline{D}) \arrow[ur] & \end{tikzcd}$$

such that the diagram is terminal among all such diagrams. It is not difficult to check that $W$ is a universal end for the functor defined above, that is, it is an end, and given an end $V$ as in the diagram, there is a unique morphism $V \longrightarrow W$ which yields the appropriate commuting diagrams.

## 4. Root stacks

The notion of a root stack is something of a generalization of the notion of an orbifold with cyclic isotropy groups over a divisor. Of course, our main interest in this construction is in the case when $X$ is a smooth complex projective variety, but giving the definition for an arbitrary $\mathbb{C}$-scheme imposes no further conceptual or technical difficulties, so we will give the definition and describe some of the basic properties in this generality. We largely follow the presentations of [Bo07] and [Cad07] here (as well as [The], [Vis08] for generalities), so we direct the reader requiring further illumination on issues raised below to these references.

**4.1. Definition and construction.** We fix a $\mathbb{C}$-scheme $X$, an invertible sheaf $L$ on $X$ and $s \in H^0(X, L)$, so that if $s$ is nonzero, it defines an effective divisor $D$ on $X$. We will also fix $r \in \mathbb{N}$. Let $\mathfrak{X} = \mathfrak{X}_{(L,r,s)}$ denote the category whose objects are quadruples

$$(f : U \longrightarrow X, N, \phi, t), \tag{4.1}$$

where $U$ is a $\mathbb{C}$-scheme, $f$ is a morphism of $\mathbb{C}$-schemes, $N$ is an invertible sheaf on $U$, $t \in H^0(U, N)$ and $\phi : N^{\otimes r} \xrightarrow{\sim} f^*L$ is an isomorphism of invertible sheaves with $\phi(t^{\otimes r}) = f^*s$. A morphism

$$(f : U \longrightarrow X, N, \phi, t) \longrightarrow (g : V \longrightarrow X, M, \psi, u)$$

consists of a pair $(k, \sigma)$, where $k : U \longrightarrow V$ is a $\mathbb{C}$-morphism making

$$\begin{tikzcd} U \arrow[rr, "k"] \arrow[dr, "f"'] & & V \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

commute and $\sigma : N \xrightarrow{\sim} k^*M$ is an isomorphism such that $\sigma(t) = k^*(u)$. Moreover, the following diagram must commute:

$$\begin{tikzcd} N^{\otimes r} \arrow[r, "\phi"] \arrow[d, "\sigma^{\otimes r}"'] & f^*L \arrow[d, "\mathrm{can}"] \\ k^*M^{\otimes r} \arrow[r, "k^*\psi"'] & k^*g^*L \end{tikzcd}$$

If

$$(g : V \longrightarrow X, M, \psi, u) \xrightarrow{(l,\tau)} (h : W \longrightarrow X, J, \rho, v)$$

is another morphism, then the composition is defined as

$$(l, \tau) \circ (k, \sigma) := (l \circ k, k^*\tau \circ \sigma), \tag{4.2}$$

using the canonical isomorphism $(l \circ k)^*J \cong k^*l^*J$.

We will often use the symbols $\mathfrak{f}$, $\mathfrak{g}$ to denote objects of $\mathfrak{X}$. If it is understood that $\mathfrak{f} \in \mathfrak{X}_U$, then by $\mathfrak{f}$ we will denote the quadruple $\mathfrak{f} = (f : U \longrightarrow X, N_{\mathfrak{f}}, \phi_{\mathfrak{f}}, t_{\mathfrak{f}})$.

The category $\mathfrak{X}$ comes with a functor $\mathfrak{X} \longrightarrow \mathfrak{Sch}/\mathbb{C}$ which simply takes $\mathfrak{f}$ to $U$ and $(k, \sigma)$ to $h$.

Proposition 4.1 [Cad07, Theorem 2.3.3]. *The morphism of categories $\mathfrak{X} \longrightarrow \mathfrak{Sch}/\mathbb{C}$ makes $\mathfrak{X}$ a Deligne–Mumford stack.*

Remark 4.2. The previous statement implies that $\mathfrak{X} \longrightarrow \mathfrak{Sch}/\mathbb{C}$ is a category fibered in groupoids. Let $\mathfrak{f} \in \mathrm{Ob}\mathfrak{X}_U$ be an object of $\mathfrak{X}$ lying over $U$ as given in (4.1) and let $g : V \longrightarrow U$ be a morphism of schemes. A choice of pullback $g^*\mathfrak{f} \in \mathrm{Ob}\mathfrak{X}_V$ can easily be described by the tuple

$$(g \circ f : V \longrightarrow X, g^*N_{\mathfrak{f}}, g^*\phi_{\mathfrak{f}}, g^*t_{\mathfrak{f}})$$

and the Cartesian arrow $g^*\mathfrak{f} \longrightarrow \mathfrak{f}$ is given by $(g, \mathbb{1}_{g^*N_{\mathfrak{f}}})$.

Example 4.3 [Cad07, Example 2.4.1]. Suppose $X = \mathrm{Spec}\, A$ is an affine scheme, $L = \mathscr{O}_X$ is the trivial bundle and $s \in H^0(X, \mathscr{O}_X) = A$ is a function. Consider $U = \mathrm{Spec}\, B$, where $B = A[t]/(t^r - s)$. Then $U$ admits an action of the group of $r$th roots of unity (more precisely, of the group scheme of the $r$th roots of unity) $\boldsymbol{\mu}_r$ of order $r$, where the induced action of $\zeta \in \boldsymbol{\mu}_r$, a generator, is given by

$$\zeta \cdot a = a, a \in A, \qquad \zeta \cdot t = \zeta^{-1}t.$$

In this case, the root stack $\mathfrak{X}_{(\mathscr{O}_X,s,r)}$ coincides with the quotient stack $[U/\boldsymbol{\mu}_r]$. Thus, as a quotient by a finite group (scheme), the map $U \longrightarrow \mathfrak{X}$ is an étale cover.

Remark 4.4. If $X$ is any $\mathbb{C}$-scheme, and $L$, $s$ are as before, we may take an open affine cover $\{X_i = \operatorname{Spec} A_i\}$ such that $L|_{X_i} \cong \mathscr{O}_{X_i}$ and $s|_{X_i}$ corresponds to $s_i \in A_i$. Then by the example above,

$$\coprod_i U_i \longrightarrow \mathfrak{X}$$

is an étale cover, where $U_i = \operatorname{Spec} A[t_i]/(t_i^r - s_i)$.

There is also a functor $\pi : \mathfrak{X} \longrightarrow \mathfrak{Sch}/X$, whose action on objects and morphisms is given by

$$\mathfrak{f} \longmapsto f : U \longrightarrow X, \quad (k, \sigma) \longmapsto k;$$

this yields a 1-morphism over $\mathfrak{Sch}/\mathbb{C}$, which we will often simply write as $\pi : \mathfrak{X} \longrightarrow X$.

**4.2. Vector bundles and differentials on a root stack.** Recall (for example [G01, Definition 2.50], [LM00, Lemme 12.2.1], [Vis89, Definition 7.18]) that a quasi-coherent sheaf $\mathscr{F}$ on $\mathfrak{X}$ consists of the data of a quasi-coherent sheaf $\mathscr{F}_{\mathfrak{f}}$ for each étale morphism $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ along with isomorphisms $\alpha_k = \alpha_k^{\mathscr{F}} : \mathscr{F}_{\mathfrak{f}} \longrightarrow k^*\mathscr{F}_{\mathfrak{g}}$ for any commutative diagram

$$\begin{tikzcd} U \arrow[rr, "k"] \arrow[dr, "\mathfrak{f}"'] & & V \arrow[dl, "\mathfrak{g}"] \\ & \mathfrak{X} & \end{tikzcd} \tag{4.3}$$

such that for a composition $U \xrightarrow{k} V \xrightarrow{h} W \longrightarrow \mathfrak{X}$ one has

$$\alpha_{h \circ k} = k^*\alpha_h \circ \alpha_k. \tag{4.4}$$

A (global) section $\mathfrak{s} \in H^0(\mathfrak{X}, \mathscr{F})$ of $\mathscr{F}$ over $\mathfrak{X}$ is the data of a global section $\mathfrak{s}_{\mathfrak{f}} \in H^0(U, \mathscr{F}_{\mathfrak{f}})$ for each étale morphism $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ such that for a diagram (4.3) as above, one has

$$\alpha_k(\mathfrak{s}_{\mathfrak{f}}) = k^*\mathfrak{s}_{\mathfrak{g}}.$$

A quasi-coherent sheaf $\mathscr{F}$ on $\mathfrak{X}$ is a subsheaf of a quasi-coherent sheaf $\mathscr{G}$ if $\mathscr{F}_{\mathfrak{f}} \subseteq \mathscr{G}_{\mathfrak{f}}$ for all étale $\mathfrak{f} : U \longrightarrow \mathfrak{X}$.

Lemma 4.5. *In the situation of Example 4.3, where $X = \operatorname{Spec} A$ is affine, $U := \operatorname{Spec} A[t]/(t^r - s)$ and $\mathfrak{X} = [U/\boldsymbol{\mu}]$, then for a quasi-coherent sheaf $\mathscr{F}$ on $\mathfrak{X}$, $\mathscr{F}_U$ admits a $\boldsymbol{\mu}_r$-action compatible with that on $U$.*

Proof. We have $U \times_{\mathfrak{X}} U \cong U \times \boldsymbol{\mu}$ and under this isomorphism, the two projection maps from $U \times_{\mathfrak{X}} U$ correspond to the maps $p_U, \lambda : U \times \boldsymbol{\mu} \longrightarrow U$, where $p_U$ is the projection onto $U$ and $\lambda$ is the action on $U$. Then the required action is defined by the composition

$$p_U^*\mathscr{F}_U \xrightarrow{\alpha_{p_U}^{-1}} \mathscr{F}_{U \times_{\mathfrak{X}} U} \xrightarrow{\alpha_\lambda} \lambda^*\mathscr{F}_U.$$

This concludes the proof. $\square$

*4.2.1. The sheaf of differentials on $\mathfrak{X}$.* The sheaf of differentials $\Omega^1_{\mathfrak{X}} = \Omega^1_{\mathfrak{X}/\mathbb{C}}$ can be defined as follows. If $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ is an étale map, then we simply set

$$\Omega^1_{\mathfrak{X},\mathfrak{f}} := \Omega^1_{U/\mathbb{C}}.$$

If we are given a diagram (4.3), then from the composition $U \xrightarrow{k} V \longrightarrow \operatorname{Spec}\mathbb{C}$, one obtains a sequence

$$0 \longrightarrow k^*\Omega_{V/\mathbb{C}} \longrightarrow \Omega_{U/\mathbb{C}} \longrightarrow \Omega_{U/V} \longrightarrow 0,$$

which is left exact [The, More on Morphisms, Ch. 33, Lemma 9.9] and whose last term is zero since $k$ is necessarily étale. This defines isomorphisms $\alpha_k$. The requirement (4.4) will be met because of the universal properties these morphisms possess.

*4.2.2. The tautological invertible sheaf on $\mathfrak{X}$.* The root stack $\mathfrak{X}$ possesses a tautological invertible sheaf $\mathscr{N}$. For an étale morphism $\mathfrak{f} : U \longrightarrow \mathfrak{X}$, we simply take

$$\mathscr{N}_{\mathfrak{f}} := N_{\mathfrak{f}}.$$

Given a diagram (4.3), one has an isomorphism $(\mathbb{1}_U, \sigma) : \mathfrak{f} \longrightarrow k^*\mathfrak{g}$ and one may take

$$\alpha^{\mathscr{N}}_k := \sigma : N_{\mathfrak{f}} \longrightarrow k^* N_{\mathfrak{g}}.$$

The expression in the second component of (4.2) implies that (4.4) is satisfied. This defines the invertible sheaf $\mathscr{N}$ on $\mathfrak{X}$. Furthermore, by definition, we also get a tautological section $\mathfrak{t}$ of $\mathscr{N}$ over $\mathfrak{X}$ by simply taking

$$\mathfrak{t}_{\mathfrak{f}} := t_{\mathfrak{f}}.$$

**4.3. Higgs fields on root stacks.** Let $X$ be as in Section 2, so that it is a smooth complex projective variety; $D$ will be a normal crossing divisor with smooth components. Let $s \in H^0(X, \mathscr{O}_X(D))$ be a section with $(s) = D$. We also fix $r \in \mathbb{N}$. In all that follows $\mathfrak{X} = \mathfrak{X}_{(\mathscr{O}_X(D),r,s)}$ will be the associated root stack as constructed in Section 4.1.

Remark 4.6. Consider the fuller situation of Section 2, where $\pi : Y \longrightarrow X$ be a Galois cover of smooth complex projective varieties and there is a divisor $\overline{D}$ on $Y$ such that $\pi^* D = r\overline{D}$. Then there is an isomorphism $\phi : \mathscr{O}_Y(\overline{D})^{\otimes r} \xrightarrow{\sim} \pi^*\mathscr{O}_X(D)$ and a section $t \in H^0(Y, \mathscr{O}_Y(\overline{D}))$ such that $\phi(t^{\otimes r}) = \pi^* s$. Therefore, the quadruple $(\pi : Y \longrightarrow X, \mathscr{O}_Y(\overline{D}), \phi, t)$ defines a morphism

$$\widehat{\pi} : Y \longrightarrow \mathfrak{X}.$$

*4.3.1. Higgs fields.* Let $\mathcal{V}$ be a vector bundle on $\mathfrak{X}$. A *Higgs field* $\Phi$ on $\mathcal{V}$ is a homomorphism $\Phi : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_{\mathfrak{X}}$. This means that for each étale morphism $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ we have a homomorphism $\Phi_{\mathfrak{f}} : \mathcal{V}_{\mathfrak{f}} \longrightarrow \mathcal{V}_{\mathfrak{f}} \otimes \Omega^1_U$ such that given a diagram (4.3), we obtain a commutative square.

$$\begin{tikzcd} \mathcal{V}_{\mathfrak{f}} \arrow[r, "\Phi_{\mathfrak{f}}"] \arrow[d, "\alpha_k^{\mathcal{V}}"'] & \mathcal{V}_{\mathfrak{f}} \otimes \Omega^1_U \arrow[d, "\alpha_k^{\mathcal{V}} \otimes \alpha_k^{\Omega^1_{\mathfrak{X}}}"] \\ k^*\mathcal{V}_{\mathfrak{g}} \arrow[r, "k^*\Phi_{\mathfrak{g}}"'] & k^*\mathcal{V}_{\mathfrak{f}} \otimes k^*\Omega^1_V \end{tikzcd} \tag{4.5}$$

THEOREM 4.7. *There is an equivalence of categories of Higgs bundles on $\mathfrak{X}$ and parabolic Higgs bundles on $X$.*

PROOF. We remark that a parabolic structure is given locally, so we may assume that $X = \operatorname{Spec}A$ is affine and that the parabolic divisor $D$ is defined by $s \in A$. Then as in Example 4.3, we may take $U = \operatorname{Spec}B$ where $B = A[t]/(t^r - s)$, so that $\mathfrak{X} = [U/\boldsymbol{\mu}]$. In this case, the map $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ is étale; we will write $f : U \longrightarrow X$ for the underlying map induced from $A \longrightarrow B$. Given a vector bundle $\mathcal{V}$, by Lemma 4.5, the bundle $\mathcal{V}_{\mathfrak{f}}$ on $U$ carries a compatible $\boldsymbol{\mu}$-action. The fact that $\Phi_{\mathfrak{f}}$ commutes with this action comes from the existence of the diagram (4.5) for the two projection morphisms $U \times_{\mathfrak{X}} U \longrightarrow U$. Thus, we are reduced to the case of Γ-bundles when $\Gamma = \boldsymbol{\mu}$, which comes from Propositions 3.3 and 3.4. □

## Acknowledgement

We are very grateful to the referee for helpful comments to improve the exposition.

## References

[Bi97] I. Biswas, 'Parabolic bundles as orbifold bundles', *Duke Math. J.* **88**(2) (1997), 305–325.

[Bo07] N. Borne, 'Fibrés paraboliques et champ des racines', *Int. Math. Res. Not.* **2007**(16) (2007), Article ID rnm049, 38 pages.

[Cad07] C. Cadman, 'Using stacks to impose tangency conditions on curves', *Amer. J. Math.* **129**(2) (2007), 405–427.

[Del70] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Mathematics, 163 (Springer, Berlin, 1970).

[EV92] H. Esnault and E. Viehweg, *Lectures on Vanishing Theorems*, DMV Seminar, 20 (Birkhäuser, Basel, 1992).

[G01] T. L. Gómez, 'Algebraic stacks', *Proc. Indian Acad. Sci. Math. Sci.* **111**(1) (2001), 1–31.

[Hi87a] N. J. Hitchin, 'The self-duality equations on a Riemann surface', *Proc. Lond. Math. Soc.* **55** (1987), 59–126.

[Hi87b] N. J. Hitchin, 'Stable bundles and integrable systems', *Duke Math. J.* **54**(1) (1987), 91–114.

[LM00] G. Laumon and L. Moret-Bailly, *Champs Algébriques* (Springer, Berlin, 2000).

[Ma98] S. Mac Lane, *Categories for the Working Mathematician*, 2nd edn. Graduate Texts in Mathematics, 5 (Springer, New York, 1998).

[MY92] M. Maruyama and K. Yokogawa, 'Moduli of parabolic stable sheaves', *Math. Ann.* **293** (1992), 77–99.

[MS80] V. Mehta and C. Seshadri, 'Moduli of vector bundles on curves with parabolic structures', *Math. Ann.* **248** (1980), 205–239.

[Na] M. Namba, *Branched Coverings and Algebraic Functions*, Pitman Research Notes in Mathematics, 161 (Longman Scientific & Technical, Harlow, 1987).

[Si88] C. T. Simpson, ‘Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization’, *J. Amer. Math. Soc.* **1** (1988), 867–918.

[The] The stacks project authors, *Stacks Project*, http://math.columbia.edu/algebraic_geometry/stacks-git.

[Vis89] A. Vistoli, ‘Intersection theory on algebraic stacks and on their moduli spaces’, *Invent. Math.* **97** (1989), 613–670.

[Vis08] A. Vistoli, Notes on Grothendieck topologies, fibrered categories and descent theory, http://homepage.sns.it/vistoli/descent.pdf, 2008.

[Yo95] K. Yokogawa, ‘Infinitesimal deformations of parabolic Higgs sheaves’, *Internat. J. Math.* **6** (1995), 125–148.

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